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Journal of Functional Analysis 202 (2003) 591–599

JOURNAL OF Functional Analysis

http://www.elsevier.com/locate/jfa

The general optimal L^p-Euclidean logarithmic Sobolev inequality by Hamilton–Jacobi equations

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Received 9 July 2002; revised 9 December 2002; accepted 11 December 2002

Communicated by L. Gross

Abstract

We prove a general optimal L^p-Euclidean logarithmic Sobolev inequality by using Prékopa–Leindler inequality and a special Hamilton–Jacobi equation. In particular we generalize the inequality proved by Del Pino and Dolbeault in (J. Funt. Anal.).

Keywords: Logarithmic Sobolev inequality; Prékopa–Leindler inequality; Hamilton–Jacobi equations; Optimal constants

1. Introduction and main results

In $\mathbb{R}^n$ the following optimal Euclidean logarithmic Sobolev inequality for the Lebesgue measure dx holds,

$$\mathbf{Ent}_{dx}(f^2) = \int f^2 \log f^2 \, dx \leqslant \frac{n}{2} \log\left(\frac{2}{\pi n e} \int |\nabla f|^2 \, dx\right)$$

for any smooth function f such that $\int f^2 dx = 1$. This inequality appears in a work of Weissler in [Wei78].

Then an L^p-version, called L^p-Euclidean logarithmic Sobolev inequality, where $1 \leqslant p \leqslant n$, is given as follows:

$$\mathbf{Ent}_{dx}(|f|^p) = \int |f|^p \log |f|^p \, dx \leqslant \frac{n}{p} \log\left(\mathscr{L}_p \int |\nabla f|^p \, dx\right), \tag{1}$$

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doi:10.1016/S0022-1236(03)00047-8

for any smooth function f such that $\int |f|^p dx = 1$ and

$$\mathcal{L}_p = \frac{p}{n}\left(\frac{p-1}{e}\right)^{p-1} \pi^{-\frac{p}{2}} \left(\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(n\frac{p-1}{p}+1)}\right)^{\frac{p}{n}}.$$

Inequality (1) for $p = 1$ was written by Ledoux in [Led96], and then by Del-Pino and Dolbeault for any $1 \leqslant p < n$ in [DPD02a].

Furthermore it is known that inequality (1) is optimal. Extremal functions for $p = 1$ were given by Beckner [Bec99]. He proved that, for $p = 1$, the extremels are the characteristic functions of balls. For $p = 2$ Carlen in [Car91] and Del Pino and Dolbeault in [DPD02a], for other values of p, prove that the extremal functions are

$$\forall x \in \mathbb{R}^n, \quad f(x) = \left(\pi^{\frac{n}{2}}\left(\frac{\sigma}{p}\right)^{n\frac{p-1}{p}} \frac{\Gamma(n\frac{p-1}{p}+1)}{\Gamma(\frac{n}{2}+1)}\right)^{-1/p} \exp\left(-\frac{1}{\sigma}|x-\overline{x}|^{\frac{p}{p-1}}\right),$$

where $\sigma > 0$ and $\overline{x} \in \mathbb{R}^n$.

The first result developed in this paper is a generalization of inequality (1). For that purpose, let $C : \mathbb{R}^n \to \mathbb{R}^+$ be an even, strictly convex function. We suppose that there exist $q > 1$ such that C satisfies the following property:

$$\forall \lambda \geqslant 0, \quad x \in \mathbb{R}^n, \quad C(\lambda x) = \lambda^q C(x). \tag{2}$$

We shall say that C is q-homogeneous. Let us note by C^*, the Legendre transform of C. It's easy to prove that in this case, C^* is also even and p-homogeneous where $1/p + 1/q = 1$. The most important example is $C(x) = ||x||^q$, where $||\cdot||$ is a norm on $\mathbb{R}^n$.

The principal result of this paper is the following.

Theorem 1.1. *Let $q > 1$, $n > 0$, and C q-homogeneous (dx the Lebesgue measure on $\mathbb{R}^n$). Suppose that $1/p + 1/q = 1$. Then for any smooth function f on $\mathbb{R}^n$ such that $\int |f|^p dx = 1$, we have*

$$\mathbf{Ent}_{dx}(|f|^p) = \int |f|^p \log(|f|^p)\, dx \leqslant \frac{n}{p} \log\left(\mathcal{L}_C \int C^*(\nabla f) dx\right), \tag{3}$$

where C^* is the Legendre transform of C and

$$\mathcal{L}_C = \frac{p^{p+1}}{ne^{p-1}(\int e^{-C(x)} dx)^{p/n}}.$$

Inequality (3) *is optimal and equality holds if for some* $b>0$ *and* $\overline{x}\in\mathbb{R}^n$

$$\forall x\in\mathbb{R}^n,\quad f(x)=a\exp(-bC(x-\overline{x})), \tag{4}$$

where $a^{-p}=\int\exp(-pbC(x-\overline{x}))dx$.

Inequality (3) *is called* L^p*-Euclidean logarithmic Sobolev inequality.*

Remark. Theorem 1.1 is a generalization of Theorem 1.1 of [DPD02a], if $1\leqslant p<n$ and $C(x)=\frac{1}{q}|x|^q$ with $1/p+1/q=1$, then inequality (3) is exactly the inequality (1) with the same constant because $C^*(x)=\frac{1}{p}|x|^p$ and it's easy to prove that

$$\frac{\mathscr{L}_C}{p}=\mathscr{L}_p.$$

The generalization concerns on the one hand the function C and on the other hand the parameter $p>1$, which is not any more restricted to the set $[1,n[$.

Also let us note that the methods used by Cordero-Erausquin et al. [CENV02] allow as well to obtain a generalization of inequality (1). They use the theorem of Brenier and McCann [Br91,McC97] to prove, by a new method, the optimal Gagliardo–Niremberg inequalities, see [DPD02b]. Then the L^p-Euclidean logarithmic Sobolev inequality appears as a limit case.

The second result gives an optimal control of Hamilton–Jacobi equations which are equivalent to Theorem 1.1. Let us first define the Hamilton–Jacobi equations and solutions $(\mathbf{Q}_t^{(C)})_{t\geqslant 0}$.

If g is a smooth function on $\mathbb{R}^n$ (for example Lipschitz), the operator $(\mathbf{Q}_t^{(C)})_{t\geqslant 0}$ is defined by the following equation:

$$\begin{cases}\mathbf{Q}_t^{(C)}g(x)=\inf_{y\in\mathbb{R}^n}\{g(y)+tC(\frac{x-y}{t})\}, & t>0,\ x\in\mathbb{R}^n,\\ \mathbf{Q}_0^{(C)}g(x)=g(x), & x\in\mathbb{R}^n.\end{cases} \tag{5}$$

We know that $v=v(x,t)=\mathbf{Q}_\mathbf{t}^{(C)}g(x)$ is the solution of the following Hamilton–Jacobi equation:

$$\begin{cases}\frac{\partial v}{\partial t}(x,t)+C^*(\nabla v(x,t))=0, & t>0,\ x\in\mathbb{R}^n,\\ v(x,0)=g(x), & x\in\mathbb{R}^n.\end{cases} \tag{6}$$

This semigroup is called the Hopf–Lax solution of Hamilton–Jacobi equations. More details about Hamilton–Jacobi equations may be found in [Bar94] and [Eva98].

We obtain the following result.

Theorem 1.2. *Let $n>0$ and C q-homogeneous. Suppose that $1/p+1/q=1$. Then for any smooth function g on $\mathbb{R}^n$, $\beta \geqslant \alpha > 0$, $t>0$ we have*

$$\left|\left|e^{\mathbf{Q}_t^{(C)}g}\right|\right|_\beta \leqslant ||e^g||_\alpha \left(\frac{\beta-\alpha}{t}\right)^{\left(\frac{n}{p}\right)\frac{\beta-\alpha}{\beta\alpha}} \frac{\alpha^{\frac{n}{\beta\alpha}\left(\frac{\alpha}{p}+\frac{\beta}{q}\right)}}{\beta^{\frac{n}{\beta\alpha}\left(\frac{\beta}{p}+\frac{\alpha}{q}\right)}} \left(\frac{1}{\int e^{-C(x)}dx}\right)^{\frac{\beta-\alpha}{\beta\alpha}}, \tag{7}$$

where $||\cdot||$ is the norm of Lebesgue measure on $\mathbb{R}^n$.

Inequality (7) *is optimal and equality holds if for some $0<\alpha\leqslant\beta$, $\overline{x}\in\mathbb{R}^n$ and $b>0$ we have*

$$\begin{cases} \forall x\in\mathbb{R}^n, & g(x) = -bC(x-\overline{x}),\\ t = \frac{\beta-\alpha}{b^{p/q}\beta}. \end{cases}$$

Furthermore, when *$\beta=\infty$ and $\alpha=1$ we obtain, for any smooth function g, the following ultracontractive bounds of $(\mathbf{Q}_t^{(C)})_{t\geqslant 0}$,*

$$\left|\left|e^{\mathbf{Q}_t^{(C)}g}\right|\right|_\infty \leqslant ||e^g||_1 \left(\frac{1}{t}\right)^{\frac{n}{p}} \frac{1}{\int e^{-C(x)}dx},$$

and equality hold if $t = 1/(b^{p/q})$ and $g(x) = -bC(x)$.

Remark. The link between Hamilton–Jacobi equations where $C(x)=|x|^2/2$ and logarithmic Sobolev inequality are given in [BGL01]. In particular, the authors prove that logarithmic Sobolev inequality are equivalent to hypercontractivity of Hamilton–Jacobi equations.

And the link between Sobolev inequality and the Hamilton–Jacobi equation are given in [Gen02]. In particular, the author proves that Sobolev inequality implies an ultracontractive estimate of Hamilton–Jacobi solutions and inequality (7) is a generalization of inequality (13) of [Gen02].

We are going to see, in the next section, the link between Theorems 1.1 and 1.2.

2. Generalization on a Riemannian manifold

Let M be a smooth complete Riemannian manifold of dimension n with Riemannian metric d. If g is a smooth function on M (for example Lipschitz), the semigroup $(\mathbf{Q}_t^{(C)})_{t\geqslant 0}$ is defined by the following equation:

$$\begin{cases} \mathbf{Q}_t^{(C)}g(x) = \inf_{y\in M}\{g(y) + tC(\frac{d(x,y)}{t})\}, & t>0, \ x\in M,\\ \mathbf{Q}_0^{(C)}g(x) = g(x), & x\in M. \end{cases} \tag{8}$$

Following the argument in the classical Euclidean case, one shows similarly that $v = v(x,t) = \mathbf{Q}_{\mathbf{t}}^{(C)} g(x)$ is a solution of the initial-value Hamilton–Jacobi problem on the manifold M,

$$\begin{cases} \frac{\partial v}{\partial t}(x,t) + C^*(\nabla v(x,t)) = 0, & t>0,\ x \in M, \\ v(x,0) = g(x), & x \in M, \end{cases} \tag{9}$$

where ∇v stands for the Riemannian length of the gradient of v for the variable x.

Now let us explain the generalization.

Theorem 2.1. *Let μ be a measure on M. We suppose that μ is absolutely continuous with respect to the standard volume element on M, and let $\Phi : \mathbb{R}^+ \to \mathbb{R}$ be a strictly increasing concave function.*

Suppose that the measure μ on M satisfies the following inequality for any smooth function f on M such that $\int |f|^p d\mu = 1$,

$$\mathbf{Ent}_\mu(|f|^p) \leqslant \Phi\left(\int C^*(\nabla f) d\mu \right). \tag{10}$$

This inequality will be called the L^p-entropy-energy inequality.

Let $c>0$ and let q_c denote a strictly increasing non-negative function satisfying the following differential equation on $[0,t_0]$, $(t_0>0)$:

$$\Phi'(cq_c^p) = p^{p-1} \frac{q_c^{2-p}}{q_c'}. \tag{11}$$

Note that, at this point $q_c(0)$ is not fixed. Then for any $c>0$, the following inequality is satisfied for any smooth function g,

$$\forall t \in [0,t_0], \quad \|e^{\mathbf{Q}_t^{(C)} g}\|_{q_c(t)} \leqslant \|e^g\|_{q_c(o)} e^{\mathscr{A}(t)}, \ \text{with } \mathscr{A}(t) = \int_{q_c(o)}^{q_c(t)} \frac{\psi(cy^p)}{y^2} dy, \tag{12}$$

where $\psi(x) = \Phi(x) - x\Phi'(x)$.

Conversely, if inequality (12) is satisfied for any $c>0$ then the measure μ satisfies the inequality (10) for any smooth function f such that $\int |f|^p = 1$.

The proof of this theorem is based on the following computation $\frac{d}{dt}\|e^{\mathbf{Q}_t g}\|_{q(t)}$, this an adaptation of Theorem 3.5 of [Gen02].

Corollary 2.2. *Let (M,d) be a n-dimensional smooth Riemannian manifold and let μ be a measure on M absolute continuous with respect to the standard volume element on M.*

Suppose that there exists $\mathscr{A}>0$, *such that* μ *satisfy the following* L^p*-Euclidean logarithmic Sobolev inequality for any function smooth function* f *on* M,

$$\mathbf{Ent}_\mu(|f|^p)\leqslant\frac{n}{p}\log\left(\mathscr{A}\int C^*(\nabla f)d\mu\right),\tag{13}$$

where $\mathscr{A}>0$. *Then for any* $\beta\geqslant\alpha>0$ *and any bounded Lipschitz function* g *we have*

$$\forall t>0,\quad ||e^{\mathbf{Q}_t^{(C)}g}||_\beta\leqslant||e^g||_\alpha\left(\frac{\beta-\alpha}{t}\right)^{\left(\frac{n}{p}\right)\frac{\beta-\alpha}{\beta\alpha}}\frac{\alpha^{\frac{n}{\beta\alpha}\left(\frac{\alpha}{p}+\frac{\beta}{q}\right)}}{\beta^{\frac{n}{\beta\alpha}\left(\frac{\beta}{p}+\frac{\alpha}{q}\right)}}\left(\frac{e^{p-1}n\mathscr{A}}{p^{p+1}}\right)^{\left(\frac{n}{p}\right)\frac{\beta-\alpha}{\beta\alpha}}.\tag{14}$$

Conversely if inequality (14) *is satisfied for any* $\beta\geqslant\alpha>0$ *and any bounded Lipschitz function* g *then the measure* μ *satisfies the* L^p*-Euclidean logarithmic Sobolev inequality* (13).

Proof. Let us use the Theorem 2.1 with

$$\Phi(x)=\frac{n}{p}\log(\mathscr{A}x).$$

Then for any $c>0$ we can solve the differential equation (11). Let

$$\begin{cases}\beta\geqslant\alpha>0,\\ c=\dfrac{(\beta-\alpha)n}{p^p\alpha\beta}.\end{cases}$$

Then the function $q_c(t)=\alpha\beta/((\alpha-\beta)t+\beta)$ satisfies the differential equation (11) on $[0,\infty[$.

Theorem 2.1 applied to the function $\Phi(x)=\frac{n}{p}\log(\mathscr{A}x)$ with $t_0=1$ proves the inequality (14) for $t=1$.

The definition of $\mathbf{Q}_t^{(C)}g$ imply the following scaling property:

$$\forall t>0,\quad \mathbf{Q}_t^{(C)}g=\frac{\mathbf{Q}_1^{(C)}(t^{q-1}g)}{t^{q-1}}.\tag{15}$$

By using the scaling property we prove inequality (14) for any $t>0$.

The proof of the converse is an adaptation of Theorem 3.1 of [Gen02]. $\square$

When $\beta=\infty$ and $\alpha=1$ we find the following corollary.

Corollary 2.3. *Suppose that μ satisfies the L^p-Euclidean logarithmic Sobolev inequality* (13) *then* $(\mathbf{Q}_t^{(C)})_{t\geqslant 0}$ *is bounded as follows*:

$$\forall t>0, \quad ||e^{\mathbf{Q}_t^{(C)}g}||_\infty \leqslant ||e^g||_1 \left(\left(\frac{e}{p}\right)^{p-1}\frac{n\mathscr{A}}{t}\right)^{\frac{n}{p}},$$

for any bounded Lipschitz function g.

Remark. This inequality is an generalization of the ultracontractive bounds on Hamilton–Jacobi solutions in $\mathbb{R}^n$, see the Corollary 2.2 of [Gen02].

3. Proof of Theorems 1.1 and 1.2 with Prékopa–Leindler inequality

Proof of Theorem 1.1. At the light of Corollary 2.2, to prove inequality (3) of Theorem 1.1, we just have to prove Theorem 1.2.

Let us now prove that functions defined by (4) are extremal. Let

$$\forall x\in\mathbb{R}^n, \quad f(x) = a\exp(-bC(x-\overline{x})),$$

where $a^{-p} = \int \exp(-pbC(x-\overline{x}))dx$, $b>0$ and $\overline{x}\in\mathbb{R}^n$.

An easy calculus prove that

$$\int C^*(\nabla f)dx = b^p\int f^p(x)((x-\overline{x})\cdot\nabla C(x-\overline{x}) - C(x-\overline{x}))dx.$$

The property (2) imply that

$$\forall x\in\mathbb{R}^n, \quad x\cdot\nabla C(x) = qC(x).$$

Then we find that

$$\int C^*(\nabla f)dx = \frac{b^{p-1}n}{p^2}.$$

A similarly calculus prove that

$$\mathbf{Ent}_{dx}(f) = -\frac{n}{q} + p\log a$$

and

$$\int e^{-C(x)}dx = \frac{(bp)^{n/q}}{a^p}.$$

Then inequality (3) is an equality. $\square$

Proof of Theorem 1.2. To prove inequality (7) we are going to use the Prékopa–Leindler inequality. Let us recall this inequality, and refer to [DG80] for a review.

Let $a, b > 0$, $a + b = 0$, and u, v, w three non-negative functions on $\mathbb{R}^n$. Assume that, for any $x, y \in \mathbb{R}^n$, we have

$$u(x)^a v(y)^b \leqslant w(ax + by), \tag{16}$$

then

$$\left(\int u(x)dx\right)^a \left(\int v(x)dx\right)^b \leqslant \int w(x)dx. \tag{17}$$

This inequality is also called the Brunn–Minkowski inequality, is a particular but equivalent case of the Prékopa–Leindler inequality.

Let $\alpha, \beta \in \mathbb{R}$ such that $0 < \alpha \leqslant \beta$ and let g a bounded Lipschitz function. Set for any $x \in \mathbb{R}^n$,

$$\begin{cases} u(x) = \exp(\beta \mathbf{Q}_1^{(C)} g(x)), \\ v(x) = \exp(-\theta C(x)), \\ w(x) = \exp\left(\alpha g\left(\dfrac{\beta}{\alpha} x\right)\right), \end{cases}$$

where $\theta = (\frac{\beta-\alpha}{\alpha})^{q-1}\beta$, and set $a = \alpha/\beta$, $b = (\beta - \alpha)/\beta$.

Then for any $x, y \in \mathbb{R}^n$ we have

$$\begin{aligned} u(x)^a v(y)^b &= \exp\left(\alpha \mathbf{Q}_1^{(C)} g(x) - \frac{\beta - \alpha}{\beta} \theta C(y)\right) \\ &\leqslant \exp\left(\alpha g(x - z) + \alpha C(z) - \frac{\beta - \alpha}{\beta} \theta C(y)\right), \end{aligned}$$

for all $z \in \mathbb{R}^n$. Let take $z = -((\beta - \alpha)/\alpha) y$ and by the definition of θ and the property (2) we prove that for all $x, y \in \mathbb{R}^n$ we have

$$u(x)^a v(y)^b \leqslant w(ax + by).$$

We obtain, using the Prékopa–Leindler inequality

$$||e^{\mathbf{Q}_1^{(C)} g}||_\beta \leqslant ||e^g||_\alpha \left(\frac{\alpha}{\beta}\right)^{\frac{n}{\alpha}} \left(\int e^{-\theta C(x)} dx\right)^{-\frac{\beta-\alpha}{\beta\alpha}}.$$

By using property (2) and by a change of variables for the Lebesgue measure we prove inequality (7) for $t = 1$. By using scaling property (15) we prove inequality (7) for all $t > 0$.

Let now prove that $g(x) = -bC(x - \overline{x})$, for $b > 0$ and $\overline{x} \in \mathbb{R}^n$ are extremal functions. It is easy to prove that

$$\mathbf{Q}_t g(x) = -\frac{b}{(1 - tb^{p/q})^{q/p}} C(x - \overline{x}),$$

for $0 \leqslant t < b^{-p/q}$.

And we prove that for $t = (\beta - \alpha)/(b^{p/q}\beta)$, inequality (7) is an equality. $\square$

Remark. Extremal functions of Theorems 1.1 and 1.2 are, of course, connected. One can deduce one of them from the other one. But regrettably, unlike the circumstance in Theorem 1.1 of [DPD02a], one does not know if all extremal functions are given by (4).

Acknowledgments

The author sincerely thanks Jean Dolbeault for interesting discussions. The author also thanks referees for helpful remarks.

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